$$[\mathcal{J}(\Delta\vec{y}'), \mathcal{J}(\Delta\vec{x}')] = \left[1 - \frac{i p_y \Delta y'}{\hbar} - \frac{p_y^2(\Delta y')^2}{\hbar^2} \cdots; 1 - \frac{i p_x \Delta x'}{\hbar} - \frac{p_x^2(\Delta x')^2}{\hbar^2} + \cdots\right]$$

$$\cong \frac{(\Delta x')(\Delta y')}{\hbar^2}[p_y, p_x] + \mathcal{O}((\Delta x)^3)$$

$$\Rightarrow \quad [p_x, p_y] = 0 \quad \text{or in general} \quad \boxed{[p_i, p_j] = 0}$$

$\Rightarrow$ p_x, p_y, p_z are mutually compatible observables.

— What is the effect of translation on a momentum eigenket?

$$\mathcal{J}(d\vec{x}')|\vec{p}'\rangle = \left(1 - \frac{i\vec{p}\cdot d\vec{x}'}{\hbar}\right)|\vec{p}'\rangle = \left(\frac{\hbar - i\vec{p}'\cdot d\vec{x}'}{\hbar}\right)|\vec{p}'\rangle$$

↖ changes the phase of $|\vec{p}'\rangle$ but that is all.

In fact $|\vec{p}'\rangle$ is an eigenket of $\mathcal{J}(d\vec{x}')$ w/ eigenvalue $\left(1 - \frac{i\vec{p}'\cdot d\vec{x}}{\hbar}\right)$. ← Why is this complex? Ans $\mathcal{J}(d\vec{x}')$ is unitary, not Hermitian.

This shouldn't be surprising since $[\vec{p}, \mathcal{J}(d\vec{x})] = 0$.

Summary of Canonical Commutation Rules:

$$[x_i, x_j] = 0 \qquad [p_i, p_j] = 0 \qquad [x_i, p_j] = i\hbar\delta_{ij}$$

P. A. M. Dirac "fundamental quantum conditions"

Algebraic Properties of commutators.

$[A, A] = 0$, $[A, B] = -[B, A]$

$[A, c] = 0$ $\quad c = c$-number

$[A + B, C] = [A, C] + [B, C]$

$[A, BC] = [A,B]C + B[A,C]$

$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ Jacobi Identity.

Wave Functions in Position and Momentum Space

Base kets $|x\rangle$; $x|x'\rangle = x'|x'\rangle$ and $\langle x''|x'\rangle = \delta(x''-x')$

normalization condition

$$|\alpha\rangle = \int dx'\, |x'\rangle\langle x'|\alpha\rangle \quad \text{expansion of a ket.}$$

$|\langle x'|\alpha\rangle|^2 dx'$ = probability for a particle to be found within dx' of x'. This means that the wavefunction for a particle can be written as:

$$\psi(x') = \langle x'|\alpha\rangle$$

the inner product $\langle\beta|\alpha\rangle = \int dx' \langle\beta|x'\rangle\langle x'|\alpha\rangle$ is

$$\langle\beta|\alpha\rangle = \int dx'\, \psi_\beta^*(x')\, \psi_\alpha(x') \quad \text{"overlap of two wavefunctions"}$$

Consider the expansion of a ket in terms of the eigenkets of A – an observable.

$$|\alpha\rangle = \sum_{a'} c_{a'}|a'\rangle = \sum_{a'} |a'\rangle\langle a'|\alpha\rangle$$

$$\langle x'|\alpha\rangle = \sum_{a'} \langle x'|a'\rangle\langle a'|\alpha\rangle \longrightarrow \psi_\alpha(x') = \sum_{a'} c_{a'}\, u_{a'}(x)$$

where $u_{a'}(x') = \langle x'|a'\rangle$

What about the matrix element $\langle\beta|A|\alpha\rangle$?

$$\langle\beta|A|\alpha\rangle = \int dx'\, dx'' \, \langle\beta|x'\rangle\langle x'|A|x''\rangle\langle x''|\alpha\rangle$$

$$= \int dx'\, dx'' \, \psi_\beta^*(x') \, \langle x'|A|x''\rangle \, \psi_\alpha(x'')$$

If the operator A is diagonal in the $|x\rangle$ basis. In particular, if A is function a function of $\underset{\uparrow\,\text{operator}}{x}$, this becomes very simple:

$$\langle x'|x^2|x''\rangle = x''^2\,\delta(x'-x'')$$

$$\langle\beta|x^2|\alpha\rangle = \int dx' \, \langle\beta|x'\rangle x'^2 \langle x'|\alpha\rangle = \int dx' \, \psi_\beta^*(x') \, x'^2 \, \psi_\alpha(x')$$

In general: $$\langle\beta|\underset{\substack{\uparrow \\ \text{operator}}}{f(x)}|\alpha\rangle = \int dx' \, \psi_\beta^*(x') \underset{\substack{\uparrow \\ \text{c-number.}}}{f(x')} \psi_\alpha(x')$$

Then what is the momentum operator in the $|x\rangle$ basis? In other words, what is its representation?

$$\left(1 - \frac{ip\Delta x'}{\hbar}\right)|\alpha\rangle = \int dx' \, \mathcal{J}(\Delta x')|x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' \, |x'+\Delta x'\rangle\langle x'|\alpha\rangle$$

$$= \int dx' \, |x'\rangle\langle x'-\Delta x'|\alpha\rangle$$

$$\Longrightarrow \quad \left(1 - \frac{i p \Delta x'}{\hbar}\right)|\alpha\rangle = \int dx' \, |x'\rangle \left(\langle x'|\alpha\rangle - \Delta x' \frac{\partial}{\partial x'} \langle x'|\alpha\rangle \right)$$

$$\Longrightarrow \quad p|\alpha\rangle = \int dx' \, |x'\rangle \left(-i\hbar \frac{\partial}{\partial x'} \langle x'|\alpha\rangle \right)$$

$\Longrightarrow$

$$\langle x|p|\alpha\rangle = -i\hbar \frac{\partial}{\partial x'} \langle x'|\alpha\rangle$$

In particular, $\langle x|p|x''\rangle = -i\hbar \dfrac{\partial}{\partial x'} \delta(x'-x'')$

and

$$\langle \beta|p|\alpha\rangle = \int dx' dx'' \langle \beta|x'\rangle \langle x'|p|x''\rangle \langle x''|\alpha\rangle$$

$$= \int dx' dx'' \, \psi_\beta^*(x') \, -i\hbar \frac{\partial}{\partial x'} \, \delta(x'-x'') \, \psi_\alpha(x'')$$

$$= \int dx' \, \psi_\beta^*(x') \left(-i\hbar \frac{\partial}{\partial x'}\right) \psi_\alpha(x')$$

Momentum Eigenkets and wavefunctions in momentum space:

$$p|p'\rangle = p'|p'\rangle \quad ; \quad \langle p'|p''\rangle = \delta(p'-p'')$$

We can define a momentum space wavefunction from the ket $|\alpha\rangle$ as follows:

$$|\alpha\rangle = \int dp' \, |p'\rangle\langle p'|\alpha\rangle \quad ; \quad \varphi_\alpha(p') = \langle p'|\alpha\rangle$$

Normalization: $\displaystyle \int dp' \langle \alpha|p'\rangle\langle p'|\alpha\rangle = \int dp' \, |\langle p'|\alpha\rangle|^2 = 1$

Relating position and momentum space representations of an eigenket

$\Rightarrow$ We need a transformation function of the form:
$\langle x' | p' \rangle$

From $\langle x' | p | \alpha \rangle = -i\hbar \frac{\partial}{\partial x'} \langle x' | \alpha \rangle$ and taking $|\alpha\rangle = |p'\rangle$

$$\langle x' | p | p' \rangle = -i\hbar \frac{\partial}{\partial x'} \langle x' | p' \rangle \quad \text{or}$$

$$\frac{\partial}{\partial x'} \langle x' | p' \rangle = \frac{ip'}{\hbar} \langle x' | p' \rangle \longleftarrow \text{Differential Equation for the function } \langle x' | p' \rangle$$

$\Rightarrow$
$$\langle x' | p' \rangle = N e^{\frac{ip'}{\hbar} x'}$$

N: constant related to normalization

$\langle x' | p' \rangle$: wavefunction of a state with definite momentum: an eigenket of the p-operator.

To determine N consider

$$\delta(x' - x'') = \int dp \langle x' | p \rangle \langle p | x'' \rangle = |N|^2 \int_{-\infty}^{\infty} dp \; e^{\frac{ip}{\hbar}(x'' - x')}$$

$$= |N|^2 \hbar \int_{-\infty}^{\infty} e^{ip(x'' - x')} dp$$

$$= 2\pi \, \delta(x'' - x') |N|^2 \hbar$$

$\Rightarrow$ $N = \dfrac{1}{\sqrt{2\pi\hbar}}$ up to a phase factor.

By convention we take the phase to be 1.

$$\Rightarrow \qquad \langle x' | p' \rangle = \frac{1}{\sqrt{2\pi\hbar}} \, e^{\frac{ip'x'}{\hbar}}$$

Now, to put it to work!

$$\langle x | \alpha \rangle = \int dp' \, \langle x | p' \rangle \langle p' | \alpha \rangle \quad \text{or}$$

$$\psi_\alpha(x) = \int_{-\infty}^{\infty} \frac{dp'}{\sqrt{2\pi\hbar}} \, e^{\frac{ip'x}{\hbar}} \, \varphi_\alpha(p') \quad \text{and}$$

$$\langle p | \alpha \rangle = \int dx' \, \langle p | x' \rangle \langle x' | \alpha \rangle \quad \Rightarrow$$

$$\varphi_\alpha(p) = \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{2\pi\hbar}} \, e^{\frac{-ipx'}{\hbar}} \, \psi_\alpha(x')$$

Our formalism naturally generates Fourier transforms.

Quantum Dynamics

Time Evolution and the Schrödinger Equation

How does a ket change in time

$|\alpha, t_0; t\rangle \qquad t > t_0$

parameter

We require $\lim_{t \to t_0} |\alpha, t_0; t\rangle = |\alpha\rangle \leftarrow$ means $|\alpha, t_0\rangle$

Time Evolution operator: $\mathcal{U}(t, t_0)$

$|\alpha, t_0; t\rangle = \mathcal{U}(t, t_0) |\alpha, t_0\rangle$

Properties of $\mathcal{U}(t, t_0)$:

(i) Unitary If $|\alpha, t_0\rangle = \sum_{a'} c_{a'} |a'\rangle$ then later

$$|\alpha; t_0\rangle = \sum_{a'} c_{a'}(t) |a'\rangle$$

We don't expect $c_{a'} = c_{a'}(t)$ for all a' i.e. no time evolution but we do expect:

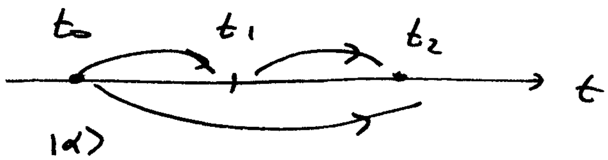
$$\sum_{a'} |c_{a'}|^2 = \sum_{a'} |c_{a'}(t)|^2$$

i.e. $\langle\alpha, t_0 | \alpha, t_0\rangle = \langle\alpha, t_0; t | \alpha, t_0; t\rangle = 1$

$\Longrightarrow$ $$\boxed{\mathcal{U}^\dagger(t,t_0)\,\mathcal{U}(t,t_0) = \mathbb{1}}$$

(ii) Composition property:

$$\mathcal{U}(t_2,t_0) = \mathcal{U}(t_2,t_1)\,\mathcal{U}(t_1,t_0); \quad t_0 < t_1 < t_2$$



(iii) Continuously connected to the identity:

$$\lim_{dt\to 0} \mathcal{U}(t_0+dt, t_0) = \mathbb{1}.$$

$\Longrightarrow$

$$\mathcal{U}(t_0+dt, t_0) = \mathbb{1} - i\Omega\, dt$$ where Ω is a Hermitian operator.

dimensions of inverse time = frequency

Thinking about $E = \hbar\omega$ or that H is the generator of time translation, we claim:

$$\Omega = H/\hbar$$

$\Longrightarrow$

$$\mathcal{U}(t_0+dt, t_0) = \mathbb{1} - \frac{iH\,dt}{\hbar}$$

Turns out we need to use the same constant $\hbar$ here as we did with $\vec{p}$ and $\vec{J}$

The Schrödinger Equation

$$\mathcal{U}(t+dt, t_0) = \mathcal{U}(t+dt, t)\,\mathcal{U}(t, t_0) = \left(1 - \frac{iH\,dt}{\hbar}\right)\mathcal{U}(t, t_0)$$

or

$$\mathcal{U}(t+dt, t_0) - \mathcal{U}(t, t_0) = \frac{-iH}{\hbar}\,dt\,\mathcal{U}(t, t_0)$$

$$\Rightarrow \quad (\star)\quad \boxed{i\hbar \frac{\partial}{\partial t}\mathcal{U}(t, t_0) = H\,\mathcal{U}(t, t_0)}$$

Schrödinger Equation for the time evolution operator.

What about the time evolution of the ket $|\alpha; t_0\rangle$?

$$i\hbar\,\partial_t\,\mathcal{U}(t, t_0)|\alpha, t_0\rangle = H\,\mathcal{U}(t, t_0)|\alpha, t_0\rangle \Rightarrow$$

$(\star\star)$ $\boxed{i\hbar\,\partial_t |\alpha, t_0; t\rangle = H|\alpha, t_0; t\rangle}$ ⟵ Schrödinger Equation for the time-dependent ket $|\alpha, t_0; t\rangle$

A formal solution for $\mathcal{U}$ in the case that H is time-independent.

Clearly, $\mathcal{U}(t, t_0) = e^{\frac{-iH(t-t_0)}{\hbar}}$ is a solution of $(\star)$ with initial condition $\mathcal{U}(t_0, t_0) = \mathbb{1}$.

What if $H = H(t)$? In particular, what if $[H(t_1), H(t_2)] \neq 0$? We will come back to this in Chapter 5.

Energy Eigenkets:

Consider the case where we know the set of base kets $\{|a\rangle\}$ which are eigenkets of the observable A that is compatible with H, i.e. $[A, H] = 0$

Then $\quad H\,|a'\rangle = E_{a'}\,|a'\rangle$

these are also energy eigenkets.

Now, expand the time evolution operator in the $\{|a\rangle\}$ basis:

$$e^{-\frac{iHt}{\hbar}} = \sum_{a'a''} |a'\rangle\langle a'|\, e^{-iHt/\hbar}\, |a''\rangle\langle a''|$$

$$e^{-iHt/\hbar} = \sum_{a'} e^{-iE_{a'}t/\hbar}\, |a'\rangle\langle a'|$$

We can now solve immediately <u>any</u> initial value problem

$|\alpha, t_0\rangle$ is given. What is $|\alpha, t_0; t\rangle$? take $t_0 = 0$ here

$$|\alpha, t\rangle = \mathcal{U}(t,0)\,|\alpha\rangle$$

$$|\alpha, t\rangle = e^{-iHt/\hbar}\,|\alpha\rangle = \sum_{a'} |a'\rangle \underbrace{\langle a'|\alpha\rangle}\, e^{-iE_{a'}t/\hbar}$$

The expansion coefficients evolve in time

Notice that the time-evolution of the eigenkets of A is particularly simple.

$$|a',t\rangle = |a'\rangle e^{-iE_{a'}t/\hbar}$$

picks up a phase modulation only.

in particular:

$$\langle a',t|A|a',t\rangle = a' = \langle a'|A|a'\rangle \quad \text{Time-independent}$$

$\Rightarrow$ $[A,H]=0 \iff$ A is a <u>constant of the motion</u>

In principle, you can repeat the above analysis with multiple compatible observables that commute with the Hamiltonian.

The Time-Dependence of Expectation Values.

Say the system starts in an eigenket of observable A. $[A,H]=0$.
Now what is the time dependence of B, an observable that may not commute with either A or H?

$$|a', t_0=0; t\rangle = \mathcal{U}(t,0)|a'\rangle$$

$$\langle B\rangle(t) = \langle a', t_0=0, t|B|a', t_0=0, t\rangle$$

$$= \langle a'|\mathcal{U}^\dagger(t,0)\, B\, \mathcal{U}(t,0)|a'\rangle$$

$$= \langle a'|e^{iE_{a'}t/\hbar} B e^{-iE_{a'}t/\hbar}|a'\rangle = \langle a'|B|a'\rangle = \langle B\rangle(t=0)$$

$\langle B\rangle$ is constant as well. We call $|a'\rangle$ a <u>stationary state</u>

If you consider a superposition of stationary states: